Product Cordial Labeling in Context of Some One Point Union Graphs

U. M. Prajapati[1] & K. K. Raval[2]
[1]Associate Professor,[2]Asisstant Professor
[1,2]St. Xavier's College, Mathematics Department, Ahmedabad, 380009, Gujarat, India.

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ABSTRACT *A graph $G = (V(G), E(G))$ is said to be a product cordial graph if there exists a function $f: V(G) \rightarrow \{0, 1\}$ with each edge uv assign the label $f(u)f(v)$, such that the number of vertices with label 0 and the number of vertices with label 1 differ atmost by1, and the number of edges with label 0 and the number of edges with label 1 differ by atmost 1. We discuss different graphs like generalized wheel graph $W(m,n)$, windmill graph $Wd(m,n)$, flower wheel graph FW_n^m, barbell graph $B(m,n)$ and graph obtained by duplication of some graph elements in generalized wheel graph are product cordial graph.*

Keywords: *Product cordial, generalized wheel graph W(m,n), windmill graph Wd(m,n), flower wheel graph FW_n^m, barbell graph B(m,n).*
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I. Introduction

We begin with simple, finite, undirected graph $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ are vertex set and edge set respectively. For all other terminology we follow Gross [6]. Now we provide brief summary of definitions and other information which are necessary for the present investigations.

Definition 2.1: A mapping $f: V(G) \rightarrow \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of Gunder f.

Definition 2.2: Let $f: V(G) \rightarrow \{0, 1\}$ and for each edge uv assign the label $|f(u) - f(v)|$, then f is said to be a cordial labeling of G if the number of vertices with label 0 and the number of vertices with label 1 differ atmost by 1, and the number of edges with label 0 and the number of edges with label 1 differ by atmost 1.

Definition 2.3: A product cordial labeling of graph G with vertex set V is a function $f : V \rightarrow \{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, the number of vertices with label 0 and the number of vertices with label 1 differ by atmost 1 and the number of edge with label 0 and the number of edge with label 1 differ by atmost 1. A graph which admits product cordial labeling is called a product cordial graph.

The notion product cordial labeling was introduced by Sundaram, Ponraj and Somasundaram [8]. They proved that following graphs are product cordial graph: tree, unicyclic graph of odd order, triangular snakes, dragons and helms. Few results of product cordial graph in the context of some graph operations on gear graph are discussed by Prajapati and Raval [7].

Definition 2.4: The neighborhood of a vertex v of a graph is the set of all vertices adjacent to v. It is denoted by $N(v)$.

Definition 2.5: Duplication of a vertex of the graph G is the graph G' obtained from G by adding a new vertex v' to G such that $N(v') = N(v)$.

Definition 2.6: Duplication of a vertex v_k by a new edge $e = v_k' v_k''$ in a graph G produces a new graph G' such that $N(v_k') = \{v_k, v_k''\}$and $N(v_k'') = \{v_k, v_k'\}$.

The notions of duplication of a vertex by a new edge and duplication of an edge by a new vertex were introduced by Vaidya and Barasara [9].

Definition 2.7: The graph $G^{(n)}$ obtained by identifying the roots of n copies of G is called the one point union of n copies of the graph G [5].

Definition 2.8: Generalized wheel graph $W(m,n)$ is a one point union of m copies of wheel graph W_n at an apex vertex.

Definition 2.9: Windmill graph $Wd(m,n)$[10] is one point union of n copies of m-complete graph K_m.The windmill graph $Wd(2,n)$ is the star graph. The windmill graph $Wd(3,n)$ is the friendship graph. The windmill graph $Wd(3,2)$ is the butterfly graph.

Benson and Lee [2] have investigated the regular windmill graphs K_m^n and determined precisely which ones are cordial for $m < 14$.

Definition 2.10: Flower wheel graph FW_m^n[3] is one point union graph of n copies of W_m connected at one common rim vertex of the wheel.

Definition 2.11: Barbell graph $B(m,n)$[1] is obtained by n copies of complete graph K_m connected by a bridge.

II. Main Results

Theorem 3.1: Generalized wheel graph $W(m,n)$ is product cordial graph if m is even, $n \geq 2$.

Proof: Let $G = W(m,n)$ be one point union of m copies of wheel graph W_n. Let the consecutive rim vertices of the i^{th} copy of $W(m,n)$ be $v_{i_1}, v_{i_2}, v_{i_3}, \dots, v_{i_n}$ and let v_0 be an apex vertex. Thus $|V(G)| = mn + 1$and $|E(G)| = 2mn$.

Define a function $f: V(G) \to \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & if\ x = v_0; \\ 1, & if\ x = v_{i_j}, 1 \leq i \leq \dfrac{m}{2}\ and\ j = 1,2,\dots,n; \\ 0, & if\ x = v_{i_j}, i > \dfrac{m}{2}\ and\ j = 1,2,\dots,n. \end{cases}$$

Thus $v_f(0) = \frac{mn}{2}, v_f(1) = \frac{mn}{2} + 1, e_f(0) = e_f(1) = mn$.

Clearly $|v_f(0) - v_f(1)| \leq 1$and $|e_f(0) - e_f(1)| \leq 1$. So G admits product cordial labeling. Hence G is a product cordial graph.

Illustration 3.1: Product cordial labeling of $W(4,4)$ is shown in the figure below.

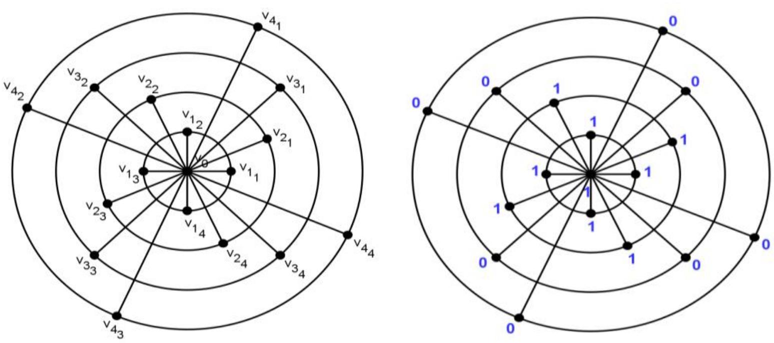

Fig. 1

Theorem 3.2: The graph G' obtained by duplication of each of the rim edges in generalized wheel graph $W(m,n)$ by a vertex is product cordial graph.

Proof: Let $G = W(m,n)$ be one point union of m copies of wheel graph W_n. Let the consecutive rim vertices of the i^{th} copy of $W(m,n)$be $v_{i_!}, v_{i_2}, v_{i_3}, \dots, v_{i_n}$ and let v_0 be an apex vertex. Thus $|V(G)| = mn + 1$and $|E(G)| = 2mn$.Let G' be the graph obtained from G by duplicating each of the rim edges by a vertex $u_{i_j}, 1 \leq i \leq m\ and 1 \leq j \leq n$.Thus $|V(G')\backslash right| = 2mn + 1$ and $|E(G^{'})| = 4mn$.Define a function $f: V(G^{'}) \to \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & if\ x = v_0; \\ 1, & if\ x = v_{i_j}, v_{i_j} \in V(G); \\ 0, & otherwise. \end{cases}$$

Thus $v_f(0) = mn$, $v_f(1) = mn + 1$, $e_f(0) = e_f(1) = 2mn$. Clearly $|v_f(0) - v_f(1)| \leq 1$and $|e_f(0) -$ *ef1≤1*. So G'admits product cordial labeling.Hence $G'$is a product cordial graph.

Illustration 3.2: Product cordial labeling of generalized wheel graph $W(2,3)$ is shown in the figure 2.

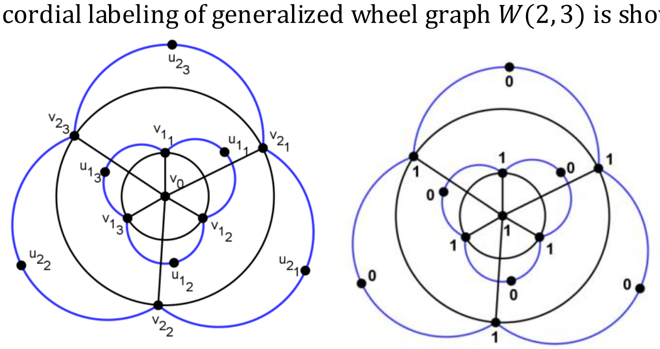

Fig. 2

Theorem 3.3: The graph G' obtained by switching of an apex vertex in generalized wheel graph $W(m,n)$ is product cordial graph for $m \geq 3$.

Proof: Let $G = W(m,n)$ be one point union of m copies of wheel graph W_n. Let the consecutive rim vertices of the i^{th} copy of $W(m,n)$ be $v_{i_1}, v_{i_2}, v_{i_3}, \dots, v_{i_n}$ and let v_0 be an apex vertex. Thus $|V(G)| = mn + 1$ and $|E(G)| = 2mn$. Let G' be the graph obtained by vertex switching of an apex vertex $v_0$in $W(m,n)$, then $|V(G')| = mn + 1$ and $|E(G')| = mn$.

- Case 1: m is even. Define a function $f: V(G^{'}) \rightarrow \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & if\ x = v_{i_j}, 1 \leq i \leq \dfrac{m}{2}, 1 \leq j \leq n; \\ 1, & if\ x = v_{i_j}, i = \dfrac{m}{2} + 1, j = 1; \\ 0, & otherwise. \end{cases}$$

Thus $v_f(0) = \frac{mn}{2}$, $v_f(1) = \frac{mn}{2} + 1$, $e_f(0) = e_f(1) = \frac{mn}{2}$. Clearly $|v_f(0) - v_f(1)| \leq 1$and $|e_f(0) - e_f(1)| \leq 1$.

- Case 2: m is odd.Define a function f: $V(G^{'}) \rightarrow \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & \text{if } x = v_{i_j}, 1 \leq i \leq \dfrac{m-1}{2}, 1 \leq j \leq n; \\ 1, & \text{if } x = v_{i_j}, i = \dfrac{m+1}{2}, 1 \leq j \leq \left\lceil \dfrac{n+1}{2} \right\rceil; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $v_f(0) = \frac{mn}{2}$, $v_f(1) = \frac{mn}{2} + 1$, $e_f(0) = \left\lceil \frac{mn}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{mn}{2} \right\rfloor$. Clearly $\left|v_f(0) - v_f(1)\right| \leq 1$and $\left|e_f(0) - e_f(1)\right| \leq 1$.

Hence from both the cases G' is product cordial graph.

Illustration 3.3: Product cordial labeling of the graph obtained by vertex switching of an apex vertex inW(2,6) andW(3,4) is shown in the figure 3.

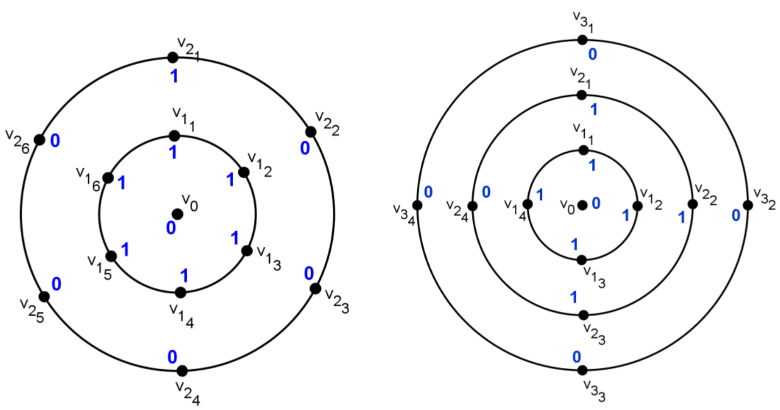

Fig. 3

Theorem 3.4: The graph G_0 obtained by duplication of each of the vertices in generalized wheel graph W(m, n) by an edge is product cordial graph.

Proof: Let $G = W(m,n)$ be one point union of m copies of wheel graph W_n. Let the consecutive rim vertices of the i^{th} copy of W(m, n) be $v_{i_1}, v_{i_2}, v_{i_3}, \dots, v_{i_n}$ and let v_0 be an apex vertex. Thus $|V(G)| = mn + 1$and $|E(G)| = 2mn$. Let G' be the graph obtained from G by duplicating each vertex v_{i_j} by an edge $u_{i_j}u_{i_j}^{'}$, $1 \leq i \leq m$ and $1 \leq j \leq n$ and duplicate v_0 by an edge $u_0u_0^{'}$ thus $V(G') = V(G) \cup \{u_0, u_0^{'}\} \cup \{u_{i_j}, u_{i_j}^{'}, 1 \leq i \leq m$ and $1 \leq j \leq n\}$and $E(G^{'}) = E(G) \cup \{v_0u_0, u_0u_0^{'}, v_0u_0^{'}\} \cup \{v_{i_j}u_{i_j}, u_{i_j}u_{i_j}^{'}, v_{i_j}u_{i_j}^{'}, 1 \leq i \leq m$ and $1 \leq j \leq n\}$.Thus $|V(G')| = 3(mn + 1)$and $|E(G')| = 5mn + 3$. We define a sequence $S: u_{1_1}, u_{1_2}, u_{1_3}, \dots, u_{1_n}, u_{2_1}, u_{2_2}, u_{2_3}, \dots, u_{2_n}, u_{m_1}, u_{m_2}, u_{m_3}, \dots, u_{m_n}$.

- Case 1: n is even. Define a function f: $V(G^{'}) \rightarrow \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & \text{if } x \in V(G); \\ 1, & \text{if } x = u_{1_1}^{'}; \\ 1, & \text{first} \dfrac{mn}{2} \text{verties of the sequence S}; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $|v_f(0) - v_f(1)| \leq 1$and $|e_f(0) - e_f(1)| \leq 1$.

- Case 2: n is odd. Define a function f: $V(G') \rightarrow \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & if x \in V(G); \\ 1, & if x = \{u_0, u_0'\}; \\ 1, & first \left\lceil \frac{mn-1}{2} \right\rceil verties of the sequence S; \\ 0, & otherwise. \end{cases}$$

Thus $|v_f(0) - v_f(1)| \leq 1$and $|e_f(0) - e_f(1)| \leq 1$.Thus, from both the above cases G' admits product cordial labeling.

Illustration 3.4: Product cordial labeling of the graph G' obtained by duplication of each of the vertices in generalized wheel graph $W(2,5)$ by an edge is shown in the figure 4.

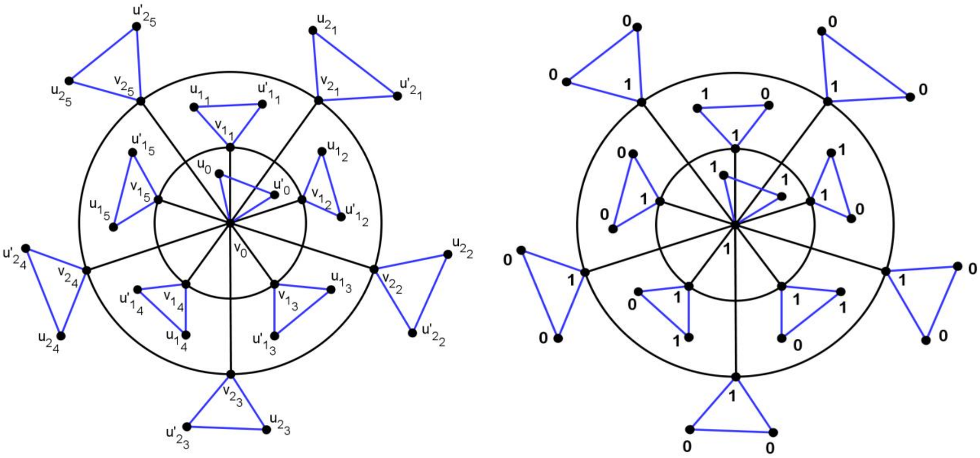

Fig. 4

Theorem 3.5: Windmill graph $Wd(m,n)$is a product cordial graph if n is even, $m \geq 4$.
Proof: Let $G = Wd(m,n)$ be one point union of n copies of m-complete graph K_m. Let v_0 be an apex vertex and let $v_{i_1}, v_{i_2}, \ldots v_{i_{m-1}}, 1 \leq i \leq n$ be the label of remaining vertices of i^{th} copy of K_m. Thus $|V(G)| = (m-1)n + 1$and$|E(G)| = \frac{nm(m-1)}{2}$. Define a function $f: V(G) \rightarrow \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & \text{if } x = v_0; \\ 1, & \text{if } x = v_{i_j}, 1 \leq i \leq \frac{n}{2}, 1 \leq j \leq m-1; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $v_f(0) = \frac{(m-1)n}{2}$, $v_f(1) = \frac{(m-1)n+1}{2}$, $e_f(0) = e_f(1) = \frac{nm(m-1)}{4}$. Clearly $|v_f(0) - v_f(1)| \leq 1$and $|e_f(0) - e_f(1)| \leq 1$. Thus $Wd(m,n)$ admits product cordial labeling, hence it is a product cordial graph, when n is even.

Illustration 3.5: Product cordial labeling of windmill graph $Wd(5,2)$ is shown in the figure 5.

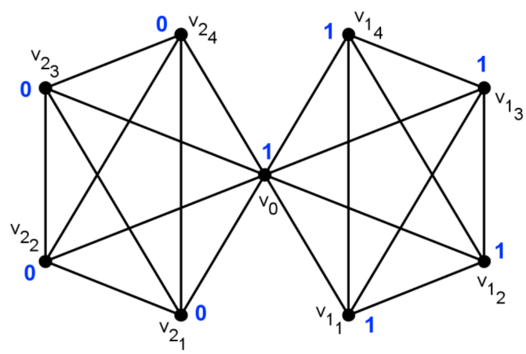

Fig. 5

Theorem 3.6: Flower wheel graph FW_m^n is a product cordial graph if and only if n is even.

Proof: Let $G = FW_m^n$ be one point union graph of n copies of W_m and the point union of the graph is any one rim vertex of the wheel.Let u be the point union vertex of the graph G, v_i be an apex vertex of i^{th} wheel and $v_{i_1}, v_{i_2}, \dots v_{i_{m-1}}, 1 \le i \le n$ be the remaining vertices of the i^{th} wheel graph W_m. Thus $|V(G)| = mn + 1$and $|E(G)| = 2mn$.

- Case 1: n is even.Define a function $f: V(G) \to \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & \text{if } x = u; \\ 1, & \text{if } x = v_i, 1 \le i \le \frac{n}{2}; \\ 1, & \text{if } x = v_{i_j}, 1 \le i \le \frac{n}{2}, 1 \le j \le m; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $v_f(1) = \frac{mn}{2} + 1$, $v_f(0) = \frac{mn}{2}$, $e_f(0) = e_f(1) = mn$. Clearly $|v_f(0) - v_f(1)| \le 1$and $|e_f(0) - e_f(1)| \le 1$.

Thus G is product cordial graph if n is even.

- Case 2: n is odd.
  - Subcase 1: m is odd. We label $\frac{mn+1}{2}$ vertices with label 0 and $\frac{mn+1}{2}$ vertices with label 1. This gives us atleast $mn + 2$ vertices with label 0 and atmost $mn - 2$ vertices with label 1.Clearly,$|e_f(0) - e_f(1)| > 1$.
  - Subcase 2: m is even. We label $\left\lfloor\frac{mn+1}{2}\right\rfloor$ vertices with 0 and $\left\lceil\frac{mn+1}{2}\right\rceil$ vertices with 1 which gives us atmost $mn - 1$ edges with label 1 and atleast $mn + 1$ edges with label 0.Clearly, $|e_f(0) - e_f(1)| > 1$.

Hence from both the above subcases G does not admit product cordial labeling.

Hence Flower wheel graph FW_m^n is a product cordial graph if and only if n is even.

Illustration 3.6: Product cordial labeling of flower wheel graph FW_6^4 is shown in the figure 6.

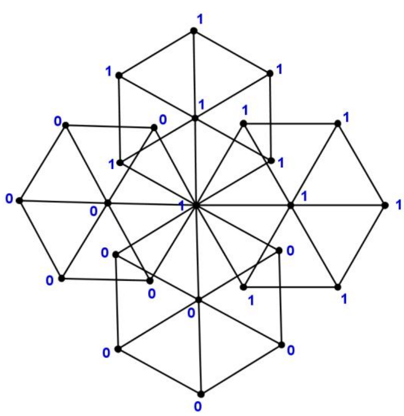

Fig. 6

Theorem 3.7: Barbell graph $B(m,n)$ is product cordial graph if $n = 3$ and for $n > 3$, $B(m,n)$ is product cordial if and only if m is even.

Proof: Let $G = B(m,n)$ be m copies of K_n connected by a bridge. Let $v_{i_1}, v_{i_2}, \dots, v_{i_n}, 1 \le i \le m$ be the consecutive vertices of the complete graph K_n. For each $i = 1,2,\dots,m-1$, join i^{th} and $(i+1)^{th}$ copy of K_n by an edge $v_{i_1}v_{i+1_1}$. Thus $|V(G)| = mn$and $|E(G)| = \frac{mn(n-1)}{2} + m - 1$.

- Case 1: m is even. Define a function $f: V(G) \to \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & \text{if } x = v_{i_j}, 1 \le i \le \frac{m}{2}, 1 \le j \le n; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $v_f(0) = v_f(1) = \frac{mn}{2}$, $e_f(0) = \frac{mn(n-1)+2m}{4}ande_f(1) = \frac{mn(n-1)+2m}{4}$. Clearly $|v_f(0) - v_f(1)| \le 1$and $|e_f(0) - e_f(1)| \le 1$.

Thus G is admits product cordial labeling if m is even.

- Case 2: m is odd.

- Subcase 1: $n = 3$. Define a function $f: V(G) \rightarrow \{0,1\}$ as follows,

$$f(x) = \begin{cases} 1, & \text{if } x = v_{i_j}, 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor, 1 \leq j \leq 3; \\ 1, & \text{if } x = v_{i_j}, i = \frac{m+1}{2}, j = 1,2; \\ 0, & \text{otherwise.} \end{cases}$$

Thus $v_f(0) = \left\lceil \frac{3m}{2} \right\rceil$, $v_f(1) = \left\lfloor \frac{3m}{2} \right\rfloor$, $e_f(0) = 2m+1$ and $e_f(1) = 2m$. Clearly $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus G admits product cordial labeling when m is odd and $n = 3$.

- Subcase 2: $n \geq 4$.

We label: first $\left(\frac{m-1}{2}\right)$ copies of K_n by label 1, last $\left(\frac{m-1}{2}\right)$ copies of K_n by label 0 and for $\left(\frac{m+1}{2}\right)^{th}$ copy of K_n we label $\left\lceil \frac{n}{2} \right\rceil$ vertices with label 1 and $\left\lfloor \frac{n}{2} \right\rfloor$ vertices with label 0. Then, we get atmost $\left(\frac{m-1}{2}\right)\left(\frac{n(n-1)}{2} + 1\right) + \left(\frac{\left(\left\lceil \frac{n}{2} \right\rceil\right)\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)}{2}\right)$ edges with label 1 and atleast $\left(\frac{m-1}{2}\right)\left(\frac{n(n-1)}{2} + 1\right) + \frac{n(n-1)}{2} - \left(\frac{\left(\left\lceil \frac{n}{2} \right\rceil\right)\left(\left\lceil \frac{n}{2} \right\rceil - 1\right)}{2}\right)$ edges with label 0. Clearly, $|e_f(0) - e_f(1)| > 1$. Thus G does not admit product cordial labeling when m is odd and $n \geq 4$.

Illustration 3.7: Product cordial labeling of barbell graph $B(4,5)$ is shown in the figure 7.

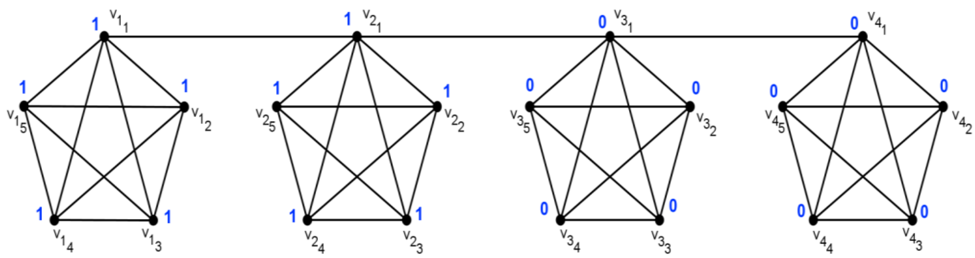

Fig. 7

Conclusion

We conclude that one point union graphs like generalized wheel graph, windmill graph, flower wheelgraph and barbell graph are product cordial graphs. We also conclude that different types of duplicationsare applied on generalized wheel graph like duplicating all rim edges with vertex and duplicating allthe vertices with an edge admits product cordial labeling. We have also switched an apex vertex ingeneralized wheel graph and the obtained graph is product cordial graph.One can investigate and study product cordial graphs with reference to various graph operations ondifferent families of graph.

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