The Double Prior Selection for the Parameter of Pareto Distribution under Type-II Censoring

Ronak M. Patel[1] & Achyut C. Patel[2]
[1]Assistant Professor, Department of Statistics, Som-Lalit College of Commerce, Navrangpura, Ahmedabad-380009
[2]Associate Professor, Department of Statistics, Smt. M. T. Dhamsania College of commerce, Kotecha Chowk, Kalawad Road, Rajkot - 360001

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ABSTRACT *This paper is concerned with the comparison of double priors assumed for the parameter of Pareto distribution for quadratic loss function under type-II censoring. We have used three different double priors viz: Jeffrey's and gamma priors, Jeffrey's and chi-square priors, gamma and chi-square priors and only gamma prior. Bayes estimation for the parameter and reliability of the distribution is done under Type-II censoring scheme. The Bayes predictive estimators for the future observation and for the remaining order statistics after the censoring period are also derived.Equal tail credible intervals are obtained in all the above situations. A simulation study is carried out to check the performance of the Bayes estimator under the double prior selection.*

Keywords: *Bayes estimation, credible interval, reliability, squared error loss function, Jeffrey's prior, gamma prior, chi-square prior.*

1. Introduction

The Pareto distribution was originated by Vilfredo Pareto (1848-1923) as a model for the distribution of income. This distribution is mostly used in business and economics applications. But now a day it is used as a model in such widely diverse areas as insurance, engineering, medical and biological sciences etc. Due to its use in life testing experiments a great attention is received for this distribution.Aggarwala and Childs (1999)studied conditional inference for the parameters of Pareto distribution under progressive censoring scheme. Charek etal (1988) and Hussain and Zimmer(2000) made comparison of estimation techniques for the three parameters Pareto distribution. Rezai et al(2010) estimated P(Y<X) for generalized Pareto distribution. Bayesian estimation under progressive censoring from generalized Pareto distribution has been considered by Mohamed et al. (2013). Details on Pareto distribution as well as areas of application can be found in Arnold(1983), Arnold and Balakrishnan (1989), Upadhyay and Peshwani(2003), AbdEllah (2003,2006), Podder et al (2004).

In life testing experiments censoring is very common. The most common censoring schemes are type-I and type-II. In type-I censoring scheme. The life test is terminated as soon as the pre-determinednumber of failures is observed. Classical as well as Bayesian estimation methods are used for making statistical inference under such types of censoring scheme. When the prior information about unknown parameters is available Bayesian method is used. Most of the work done in the area of Bayesian estimation is based on the single prior distribution for the unknown parameters but sometimes we have different information (more than one prior distribution)available for the unknown parameter of the model under study. In such case it is advisable to include all the prior information in the Bayesian estimation. Haq and Assam (2009) have considered double prior selection in the parameters of Poisson distribution. Patel and Patel(2016(a,b),2017) have considered double prior distributionsfor estimating the parameter and other reliability characteristics in case of Rayleigh, power function and exponential life time models under type-II censoring scheme.

In this paper we have considered the following three different double priors and the results based on such double priors are compared with the results based on single prior distribution.

(i) Jeffrey's prior and gamma prior
(ii) Jeffrey's prior and chi-square prior
(iii) gamma prior and chi-square prior
(iv) only gamma prior.

The rest of this paper is organized as follows. In Section 2, we describe development of posterior distribution under the above mentioned priors based on type-II censored sample. Bayes estimates of the parameter θ and reliability at time t are derived along with their equal tail credible intervals in Section 3.

Section 4 covers Bayes predictive estimation and construction of equal tail credible interval for the estimator. In Section 5 Bayes predictive estimation for the remaining (n-r) ordered failure times truncated at $X_{(r)}$ is carried out along with equal tail credible interval. A simulation study is carried out to compare the performance of the estimators under the above different prior distributions. The results obtained are demonstrated in Section 6.

2. The posterior distribution of θ under different prior distributions

A random variable X is said to have Pareto distribution if its probability density function (pdf) is given by,

$$f(x,\theta)=\theta x^{-\theta-1},\ x\geq 1,\theta>0 \quad (2.1)$$

Its cumulative distribution function $F(x,\theta)$ and reliability function $R(t)$ at time t are given as,

$$F(x,\theta)=1-x^{-\theta},\ x\geq 1,\theta>0 \quad (2.2)$$

and,

$$R(t)=t^{-\theta},\ t\geq 1,\theta>0 \quad (2.3)$$

θ is called the shape parameter of the Pareto distribution. Let n items are placed on a life test and the test is terminated after the r-th failure, $1\leq r\leq n$, r is predetermined fixed integer. Consider $x_{(1)},x_{(2)},\ldots,x_{(i)},\ldots,x_{(r)}$ are the r observed ordered failure times during the test. Here we assume that the failures are not replaced and the test is continued with the remaining items on the test. Such censoring scheme is called type-II censoring without replacement scheme.

The likelihood function under this censoring scheme is given by,

$$L(\underline{x},\theta)\propto\prod_{i=1}^{r}f\left(x_{(i)},\theta\right)\left[1-F\left(x_{(r)},\theta\right)\right]^{n-r}$$

Using (2.1) and (2.2) it reduces to

$$L=L(\underline{x},\theta)\propto\theta^{r}\left(\prod_{i=1}^{r}x(i)\right)^{-\theta-1}x(r)^{-\theta(n-r)} \quad (2.4)$$

2.1Jeffrey's and gamma double prior

Let us consider

$$P_{11}(\theta)=\frac{1}{\theta},\theta>0 \quad (2.5)$$

asJeffrey's priorand

$$P_{12}(\theta)=\frac{b_1^{a_1}}{\Gamma a_1}\theta^{a_1-1}e^{-b_1\theta};\theta>0,a_1>0,b_1>0 \quad (2.6)$$

as gamma prior for parameter θ. Combining (2.5) and (2.6), the double prior distribution for θ can be defined as

$$P_1(\theta)\propto P_{11}(\theta)P_{12}(\theta)=\theta^{a_1-2}e^{-b_1\theta};\theta>0,a_1>0,b_1>0 \quad (2.7)$$

Which is gamma distribution G(α_1,β_1) where, $\alpha_1=a_1-1$ and $\beta_1=b_1$.

2.2 Jeffrey's and chi-square double prior

Let us consider

$$P_{21}(\theta)=\frac{1}{\theta},\theta>0 \quad (2.8)$$

as Jeffrey's prior and,

$$P_{22}(\theta)=\frac{e^{-\frac{\theta}{2}}\theta^{\frac{a_2}{2}-1}}{2^{\frac{a_2}{2}}\Gamma\frac{a_2}{2}}e^{-b_1\theta};\theta>0,a_2>0 \quad (2.9)$$

as chi-square prior for parameter θ.

Combining (2.8) and (2.9), the double prior distribution for θ can be defined as,

$$P_2(\theta) \propto P_{21}(\theta) P_{22}(\theta) = \theta^{\frac{a_2}{2}-2} e^{-\frac{\theta}{2}}; \theta > 0 \quad (2.10)$$

which is a gamma distribution G(α_2, β_2) where, $\alpha_2 = \frac{a_2}{2} - 1$ and $\beta_2 = \frac{1}{2}$.

2.3 Gamma and Chi-square double prior

Let us consider $P_{31}(\theta) = \frac{b_1^{a_1}}{\Gamma a_1} \theta^{a_1 - 1} e^{-b_1\theta}; \theta > 0, a_1 > 0, b_1 > 0$ (2.11)

as gamma prior and

$$P_{32}(\theta) = \frac{e^{-\frac{\theta}{2}} \theta^{\frac{a_2}{2}-1}}{2^{\frac{a_2}{2}} \Gamma \frac{a_2}{2}} e^{-b_1\theta}; \theta > 0, a_2 > 0 \quad (2.12)$$

as chi-square prior for parameter θ.

Combining (2.11) and (2.12), the double prior distribution for θ can be defined as,

$$P_3(\theta) \propto P_{31}(\theta) P_{32}(\theta) = \theta^{a_1 + \frac{a_2}{2} - 2} e^{-\theta\left(b_1 + \frac{1}{2}\right)}; \theta > 0 \quad (2.13)$$

Which is a gamma distribution G(α_3, β_3) where, $\alpha_3 = a_1 + \frac{a_2}{2} - 1$ and $\beta_3 = b_1 + \frac{1}{2}$.

2.4 Only gamma prior

Here we consider only single prior distribution for the parameter θ as gamma prior given by,

$$P_4(\theta) \propto \theta^{a_1 - 1} e^{-b_1\theta}; \theta > 0 \quad (2.14)$$

which is a gamma distribution G (α_4, β_4) where, $\alpha_4 = a_1$ and $\beta_4 = b_1$.

Thus the double prior distribution in the i-thcase is given by

$$P_i(\theta) = \frac{\beta_i^{\alpha_i}}{\Gamma \alpha_i} \theta^{\alpha_i - 1} e^{-\beta_i \theta}; i = 1,2,3,4. \quad (2.15)$$

The posterior distribution of θ for given $\underline{x}$ in case of double prior distribution $P_i(\theta)$ can be obtained as,

$$\Pi_i(\theta \mid \underline{x}) \propto L(\underline{x}, \theta) P_i(\theta)$$

$$= \theta^r \left(\prod_{i=1}^{r} x_{(i)} \right)^{-\theta - 1} x_{(r)}^{-\theta(n-r)} \theta^{\alpha_i - 1} e^{-\beta_i \theta}$$

$$= \theta^{r + \alpha_i - 1} e^{-(\beta_i + k_r)\theta}$$

where

$$k_r = \sum_{i=1}^{r-1} \log x_{(i)} + (n - r + 1) \log x_{(r)}$$

Hence

$$\Pi_i(\theta \mid \underline{x}) = \frac{\theta^{r + \alpha_i - 1} e^{-(\beta_i + k_r)\theta} (\beta_i + k_r)^{r + \alpha_i}}{\Gamma r + \alpha_i} \quad (2.16)$$

Thus posterior distribution for i-thcase is also gamma G (c_i, d_i) where, $c_i = \alpha_i + r$ and $d_i = \beta_i + k_r ; i = 1,2,3,4$.

3. Bayes estimate of θ and reliability R(t) at time t

3.A Bayes estimate of θ under squared error loss function

$$\hat{\theta} = E_{\Pi_i}(\theta \mid \underline{x})$$

$$= \int_0^\infty \theta \Pi_i (\theta \mid \underline{x}) d\theta$$

$$= \int_0^\infty \theta \frac{d_i^{c_i}}{\Gamma c_i} \theta^{c_i - 1} e^{-d_i \theta} d\theta$$

$$= \frac{c_i}{d_i} \quad (3.1)$$

Bayes estimate of reliability $R(t) = t^{-\theta}$ at time t under square error loss function is given as

$$R(\hat{t}) = E_{\Pi_i}(R(t) \mid \underline{x})$$

$$= \int_0^\infty t^{-\theta} \frac{d_i^{c_i}}{\Gamma c_i} \theta^{c_i - 1} e^{-d_i \theta} d\theta \quad (3.2)$$

$$= \left(\frac{d_i}{d_i + \log t} \right)^{c_i}$$

3.B Bayes equal tail credible interval for θ

An interval $[I_{1i}, I_{2i}]$ is said to be the Bayes equal tail (1-γ)100% credible interval of θ if

$$\int_0^{I_{1i}} \Pi_i (\theta \mid \underline{x}) d\theta = \frac{r}{2} = \int_{I_{2i}}^\infty \Pi_i (\theta \mid \underline{x}) d\theta ; 0 < \gamma < 1$$

The value of I_{1i} can be obtained by solving the equation

$$\int_0^{I_{1i}} \Pi_i (\theta \mid \underline{x}) d\theta = \frac{\gamma}{2}$$

$$\Rightarrow \int_0^{I_{1i}} \frac{d_i^{c_i}}{\Gamma c_i} \theta^{c_i - 1} e^{-d_i \theta} d\theta = \frac{\gamma}{2}$$

$$\Rightarrow \int_0^{d_1 I_{1i}} \frac{w^{c_i - 1} e^{-w}}{\Gamma c_i} dw = \frac{\gamma}{2}$$

$$\Rightarrow G(c_i, 1, d_i I_{1i}) = \frac{\gamma}{2} \quad (3.3)$$

where $G(c_i, 1, d_i I_{1i}) = \int_0^{d_1 I_{1i}} \frac{w^{c_i - 1} e^{-w}}{\Gamma c_i} dw$

is incomplete gamma interval. Similarly to get the value of I_{2i}, for given value of r, we solve the equation

$$\int_{I_{2i}}^\infty \Pi_i (\theta \mid \underline{x}) d\theta = \frac{\gamma}{2}$$

$$\Rightarrow \int_0^{I_{2i}} \Pi_i(\theta \mid \underline{x}) d\theta = 1 - \frac{\gamma}{2}$$

$$\Rightarrow G(c_i, 1, d_i I_{2i}) = 1 - \frac{\gamma}{2} \quad (3.4)$$

3.C Bayes equal tail credible interval for $R(t) = t^{-\theta}, t \geq 1.$

Using the credible interval of θ, credible interval for R(t) can be obtained as follows.

$$P(I_{1i} \leq \theta \leq I_{2i}) = \frac{\gamma}{2}$$

$$\Rightarrow P\left(t^{-I_{1i}} \leq t^{-\theta} \leq t^{-I_{2i}}\right) = \frac{\gamma}{2} \quad (3.5)$$

$$\Rightarrow [h_{1i}, h_{2i}] = \left[t^{-I_{2i}}, t^{-I_{1i}}\right]$$

Which is the equal tail credible interval in R(t) at given time t.

4. Bayes predictive estimation and (1-γ)100% equal tail credible interval for future observation

Let Z_i be a future observation which has already survived time $X_{(r)}$. Let $w_i = z_i - x_{(r)}$. According to Hawlader and Hossian (1995) given the data $\underline{x}$, the conditional joint pdf of W_i and θ is given by,

$$h_i(w_i, \theta \mid \underline{x}) = f(w_i \mid \theta)\Pi_i(\theta \mid \underline{x})$$

$$= \frac{\theta w_i^{-\theta - 1} d_i^{c_i} \theta^{c_i - 1} e^{-d_i \theta}}{\Gamma c_i}$$

$$= \frac{\theta^{c_i}}{w_i} e^{-\theta(\log w_i + d_i)} \frac{d_i^{c_i}}{\Gamma c_i}$$

Integrating out θ, the predictive density of w_i will be,

$$P_i(w_i \mid \underline{x}) = \int_0^{\infty} h_i(w_i, \theta \mid \underline{x}) d\theta$$

$$= \int_0^{\infty} \frac{\theta^{c_i}}{w_i} e^{-\theta(\log w_i + d_i)} \frac{d_i^{c_i}}{\Gamma c_i} d\theta$$

$$= \frac{c_i d_i^{c_i}}{w_i} \frac{1}{(\log w_i + d_i)^{c_i + 1}}, w_i > 1 \quad (4.1)$$

Bayes estimation of W_i under squared error loss function is given by,

$$w_i^* = E_{P_i}(w_i \mid \underline{x}) = c_i d_i^{c_i} \int_1^{\infty} \frac{1}{(\log w_i + d_i)^{c_i + 1}} dw_i$$

Taking $y_i = \frac{\log w_i}{d_i}$ we get

$$w_i^* = c_i \int_1^{\infty} \frac{e^{y_i d_i}}{(1 + y_i)^{c_i + 1}} dy_i \quad (4.2)$$

which can be evaluated by numerical integration.

Hence Bayes predictive estimation for future observation is given by

$z_i^* = w_i^* + x_{(r)}$ (4.3)

Now we obtain (1-γ)100% equal tail credible interval $[h_{1i}, h_{2i}]$ for w_i . The limits h_{1i} and h_{2i} can be obtained by solving the equation as

$$\int_1^{h_{1i}} P_i(w_i \mid \underline{x})d\ w_i = \frac{\gamma}{2} = \int_{h_{2i}}^{\infty} P_i(w_i \mid \underline{x})d\ w_i \quad (4.4)$$

Consider the equation

$$\int_1^{h_{1i}} P_i(w_i \mid \underline{x})dw_i = \frac{\gamma}{2}$$

Using the predictive density $P_i(w_i \mid \underline{x})$ given in (4.1), we have

$$\int_1^{h_{1i}} \frac{c_i d_i^{c_i}}{w_i} \frac{1}{(\log w_i + d_i)^{c_i+1}} dw_i = \frac{\gamma}{2}$$

$$\int_{h_{1i}}^{\infty} \frac{c_i d_i^{c_i}}{w_i} \frac{1}{(\log w_i + d_i)^{c_i+1}} dw_i = 1 - \frac{\gamma}{2}$$

Let $t_i = \log w_i + d_i \Rightarrow \frac{dt_i}{dw_i} = \frac{1}{w_i}$, then we have

$$\Rightarrow c_i d_i^{c_i} \int_{\log h_{1i} + d_i}^{\infty} \frac{1}{t_i^{c_i+1}} dt_i = 1 - \frac{\gamma}{2}$$

$$\Rightarrow c_i d_i^{c_i} \left[\frac{t_i^{-c_i}}{-c_i} \right]_{\log h_{1i} + d_i}^{\infty} = 1 - \frac{\gamma}{2}$$

$$\Rightarrow d_i^{c_i} \left[(\log h_{1i} + d_i)^{-c_i} \right] = 1 - \frac{r}{2}$$

$$\Rightarrow \left(\frac{d_i}{\log h_{1i} + d_i} \right) = \left(1 - \frac{\gamma}{2} \right)^{\frac{1}{c_i}}$$

$$\Rightarrow \frac{d_i}{\left(1 - \frac{\gamma}{2}\right)^{\frac{1}{c_i}}} = \log h_{1i} + d_i$$

$$\Rightarrow h_{1i} = \exp\left(\frac{d_i}{\left(1 - \frac{\gamma}{2}\right)^{\frac{1}{c_i}}} - d_i \right) \quad (4.5)$$

Similarly solving the equation $\int_{h_{2i}}^{\infty} P_i(w_i \mid \underline{x})dw_i = \frac{\gamma}{2}$ we get

$$h_{2_i} = \exp\left(\frac{d_i}{\left(\frac{\gamma}{2}\right)^{\frac{1}{c_i}}} - d_i\right) \quad (4.6)$$

Using (4.5) and (4.6) (1-γ)100% credible interval for future observation which has already survived time $x_{(r)}$, can be obtained as

$$\left[x_{(r)} + h_{1i},\, x_{(r)} + h_{2i}\right]$$

5. Bayes predictive estimator for the remaining (n-r) order statistics truncated at $x_{(r)}$ and their (1-γ)100% equal tail credible interval

In this section we derived estimate of $X_{(s)i}$, $r+1 \le s \le n$, the failure time of s[th] - unit to fail on the basis of from r ordered failure times using the method given by Hawlader and Hossain (1995). The conditional pdf of $u_i = x_{(s)i} - x_{(r)}$ from a probability density function truncatedat $x_{(r)}$ is given by,

$$f(u_i \mid \theta) = \frac{F(u_i)^{s-r-1}(1-F(u_i))^{n-s} f(u_i)}{\beta_{(s-r,n-s+1)}}, u_i > 1$$

Using the pdf and cdf given in (2.1) and (2.2), $f(u_i \mid \theta)$ becomes

$$f(u_i \mid \theta) = \frac{\theta (u_i)^{-\theta(n-s+1)-1}\left(1-u_i^{-\theta}\right)^{s-r-1}}{\beta_{(s-r,n-s+1)}}, u_i > 1$$

Then for given the data $\underline{x}$, the conditional joint pdf of u_i and θ is given by,

$$f_i(u_i, \theta \mid \underline{x}) = f(u_i \mid \theta)\Pi_i(\theta \mid \underline{x})$$

$$= \theta^{c_i}(u_i)^{-\theta(n-s+1)-1}\left(1-u_i^{-\theta}\right)^{s-r-1} e^{-d_i\theta}\frac{d_i^{c_i}}{\Gamma c_i}; u_i > 1$$

Integrating out θ, the predictive density of u_i is given by,

$$P_i(u_i, \theta \mid \underline{x}) = \int_0^\infty \theta^{c_i}(u_i)^{-\theta(n-s+1)-1}\left(1-u_i^{-\theta}\right)^{s-r-1} e^{-d_i\theta} d\theta \frac{d_i^{c_i}}{\Gamma c_i}$$

$$= \frac{d_i^{c_i}}{u_i \Gamma c_i}\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^{s-r-1-j}\int_0^\infty \theta^{c_i} e^{-\theta\{n-r-j)\log u_i + d_i\}} d\theta$$

$$= \frac{c_i d_i^{c_i}}{u_i}\sum_{j=0}^{s-r-1}\frac{\sum_{j=0}^{s-r-1}\binom{s-r-1}{j}(-1)^{s-r-1-j}}{\{(n-r-j)\log u_i + d_i\}^{c_i+1}}; \; u_i > 1 \quad (5.1)$$

Under squared error loss function, Bayes predictive estimation of u_i is given by,

$$u_i^* = E(u_i \mid \underline{x})$$

$$= \int_1^{\infty} u_i P_i (u_i \mid \underline{x}) du_i$$

$$= \int_1^{\infty} u_i P_i (u_i \mid \underline{x}) du_i$$

$$= c_i d_i^{c_i} \int_1^{\infty} \frac{\sum_{j=0}^{s-r-1} \binom{s-r-1}{j} (-1)^{s-r-1-j}}{\{(n-r-j)\log u_i + d_i\}^{c_i+1}} du_i$$

Let $y_i = \left(\frac{n-r-j}{d_i}\right) \log u_i$ then

u_i^* reduces to

$$u_i^* = \frac{c_i}{d_i} \sum_{j=0}^{s-r-1} \binom{s-r-1}{j} (-1)^{s-r-1-j} \frac{n-r-j}{d_i} \int_0^{\infty} e^{\frac{w_i d_i}{n-r-j}} \frac{1}{(w_i+1)^{c_i+1}} dw_i \quad (5.2)$$

Putting s = r+1, we get predictive life time for (r+1)th failure as

$$x_{(r+1)} = x_{(r)} + u_i^* \quad (5.3)$$

where

$$u_i^* = \frac{c_i (n-r)}{d_i^2} \int_0^{\infty} e^{\left(\frac{d_i}{n-r}\right) w_i} \frac{1}{(w_i+1)^{c_i+1}} dw_i \quad (5.4)$$

Now we construct (1-γ)100% equal tail credible interval for u_i, denoted as $[H_{1i}, H_{2i}]$, by solving the equations

$$\int_1^{H_{1i}} P_i (u_i \mid \underline{x}) du_i = \frac{\gamma}{2} = \int_{H_{2i}}^{\infty} P_i (u_i \mid \underline{x}) du_i$$

Let us consider the equation

$$\int_1^{H_{1i}} P_i (u_i \mid \underline{x}) du_i = \frac{\gamma}{2},$$

then using (5.1) we get,

$$\Rightarrow \int_1^{H_{1i}} \frac{c_i d_i^{c_i}}{u_i} \sum_{j=0}^{s-r-1} \frac{\sum_{j=0}^{s-r-1} \binom{s-r-1}{j} (-1)^{s-r-1-j}}{\{(n-r-j)\log u_i + d_i\}^{c_i+1}} du_i = \frac{\gamma}{2}$$

Taking $w_i = \left(\frac{n-r-j}{d_i}\right) \log u_i$, we have

$$\Rightarrow c_i \sum_{j=0}^{s-r-1} \frac{\binom{s-r-1}{j} (-1)^{s-r-1-j}}{n-r-j} \int_0^{\frac{(n-r-j}{d_i} loh H_{1i}} \frac{1}{(w_i+1)^{c_i+1}} dw_i = \frac{\gamma}{2}$$

$$\Rightarrow \sum_{j=0}^{s-r-1} \binom{s-r-1}{j} (-1)^{s-r-1-j} \left(\frac{\log H_{1i}}{d_i} \right) = \frac{\gamma}{2}$$

$$\Rightarrow H_{1i} = \exp\left(\frac{\left(\frac{\gamma}{2}\right) d_i}{\sum_{j=0}^{s-r-1} \binom{s-r-1}{j} (-1)^{s-r-j}} \right) \quad (5.5)$$

Similarly by solving the equation

$$\int_{H_{2i}}^{\infty} P_i(u_i \mid \underline{x}) du_i = \frac{\gamma}{2} \text{ we get}$$

$$H_{2i} = \exp\left(\frac{\left(1-\frac{\gamma}{2}\right) d_i}{\sum_{j=0}^{s-r-1} \binom{s-r-1}{j} (-1)^{s-r-j}} \right) \quad (5.6)$$

Particularly for s = r+1, the (1-γ)100% equal tail credible interval for failure time of (r+1)th failure is given by,

$$\left[x_{(r)} + H_{1i}, x_{(r)} + H_{2i} \right] \quad (5.7)$$

where $H_{1i} = \exp\left[\left(-\frac{\gamma}{2} \right) d_i \right]$

and $H_{1i} = \exp\left[-\left(1 - \frac{\gamma}{2} \right) d_i \right]$ (5.8)

7. Simulation study

A Monte Carlo simulation study is carried out to compare the performance of the Bayes estimators under different joint priors and single prior. To generate 1000 Type-II censored samples the value of the parameter θ is considered as 8 and the values of the hyper parameters for all joint and single priors are considered as $a_i = 10$ and $b_i = 2$, i=1,2,34.. The reliability is calculated at time t = 1.2. Simulation is done for different values of sample size (n) and of fixed censored value (r) like: (n, r) = (20, 10), and (20, 15). In each case Bayes estimates of θ, R(t), future observation z^* and (r+1)th ordered failure time $X_{(r+1)}$ are obtained. Their mean square errors (MSE) and Bayes equal tail credible intervals are also obtained. The first, second and third values in each cell of columns third and fourth of Tables 2 to 5 denote the Bayes estimate, MSE and credible intervals.

Table 1. Bayes estimates, MSE and Credible intervals for θ and R(t) for n = 20

Joint priors	r	θ	R(t)
Jeffery's - Gamma	10	5.971403	0.349131
		4.518776	0.015084
		(3.59518, 8.940692)	(0.198906, 0.520462)
	15	6.344289	0.325466
		3.194908	0.010118
		(4.064911, 9.122907)	(0.192513, 0.478062)
Jeffery's - Chi square	10	8.484356	0.240116
		3.118987	0.004327
		(4.63848, 13.47219)	(0.096194, 0.435358)
	15	8.424775	0.235712

		2.339664	0.003364
		5.072270, 12.614010)	(0.108464, 0.401767)
Gamma - Chi square	10	6.509814	0.315772
		2.582011	0.008042
		(4.170966, 9.360922)	(0.183757, 0.468613)
	15	6.763179	0.300858
		1.934768	0.00581633
		(4.529415, 9.437625)	(0.181330, 0.439212)
Only gamma	10	6.285688	0.330431
		3.386039	0.011065
		(0.3.839468, 9.325091)	(0.185692, 0.497961)
	15	6.608634	0.310688
		2.428009	0.007575
		(4.276766, 9.439863)	(0.181908, 0.460102)

Table 2. Bayes estimates and Credible intervals for Z^* and $x_{(r+1)}$ for n = 20

Joint priors	r	Z^*	$x_{(r+1)}$
Jeffery's - Gamma	10	1.263673	1.150158
		0.016826	0.000287
		(2.089197, 3.084421)	(1.130727, 2.007596)
	15	1.238041	1.182352
		0.003380	0.002314
		(2.178111, 3.067604)	(1.199955, 2.082859)
Jeffery's - Chi square	10	1.394388	1.792484
		0.004763	0.006190
		(2.088012, 2.774353)	(1.282741,2.042854)
	15	1.300329	1.242940
		0.003630	0.001028
		(2.177171, 2.827669)	(1.285832, 2.117587)
Gamma - Chi square	10	1.146492	1.100279
		0.001296	0.0006824
		(2.088829, 2.943445)	(1.113043, 1.996133)
	15	1.196499	1.176049
		0.002752	0.002420
		(2.177849, 2.975700)	(1.189964, 2.071570)
Only gamma	10	1.219631	1.136838
		0.002480	0.000368
		(2.088981, 3.009688)	(1.130727, 2.007596)
	15	1.224428	1.180591
		0.003161	0.002342
		(2.177948, 3.016110)	(1.199955, 2.082859)

8. Conclusions

8-A. Comparison of priors based on the MSE and credible interval of θ.

From the third column of Tables 1 it is observed that the values of the MSE of theBayes estimator of parameter θ is smaller in case of Gamma - Chi square joint prior and then followed by Jeffery's - Chi square, only gamma and Jeffery's - Gamma priors in all the values of n and r considered here . Length of its credible interval is smallest also in Gamma - Chi square joint prior and then followed by Jeffery's - Gamma, Only gamma and Jeffery's - Chi square.

8-B. Comparison of priors based on the MSE andcredibleintervalof R(t) .

From the fourth column of Tables 1 it is observed that for all values of n and r considered here the values of the MSE of the Bayes estimator of R(t) is smaller in case of the joint prior Jeffery's - Chi square and then followed by Gamma - Chi square, Only gamma and Jeffery's - Gammapriors.

Gamma - Chi square joint prior generates the minimum length of the credible intervals of R(t) and the followed by Only gamma, Jeffery's - Gamma and Jeffery's - Chi square in case of all the values of n and r considered here.

8-C. Comparison of priors based on MSE and credible interval of future predicted value.
From the third column of Tables 2 we find that MSE is minimum in case of gamma -chi square prior and then followed by only gamma,Jeffery's - Chi square and Jeffery's - Gammapriors in case of all the values of n and r considered here while the length of the credible interval is minimum in Jeffery's - Chi square and then followed by Gamma - Chi square, Only gamma and Jeffery's - Gamma.

8-D. Comparison based on the MSE and credible interval of next ordered failure time $X_{(r+1)}$.
From the fourth column of the Tables 2 we observed minimum MSE in case of Jeffery's - Gamma joint prior and then followed by only gamma, Gamma - Chi square and Jeffery's - Chi square priors when r = 10 whereas for r = 15 minimum MSE is generated by Jeffery's - Chi square and then followed by Jeffery's - Gamma, Only gamma and gamma – chi square joint prior where as other priors give erratic effect.The length of credible interval of $X_{(r+1)}$ is smallest in case of Jeffery's - Chi square and then followed by Jeffery's - Gamma, only gamma and Gamma - Chi square.
Thus we observed that gamma – chisquarejoint prior performs well for estimating the parameter θ, R(t)as well as for estimating future predicted valueZ^*compared to the other single and joint priors considered in this study.

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