On Application of Fixed Point Theorem for Solving Initial Value Problems and Integral Equations

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Abstract

In this paper we have discussed the solution of integral equation and initial value problem via Fixed point method. These equations sometimes can not be solved explicitly but easy to solve by numerical techniques, approximation methods.

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1 Introduction:

The fixed point theory is a vast and interdisciplinary area for many mathematical domains. It has most rapidly growing applications to integral equations of many types. Ran and Reunnings[1] initiated the fixed point in metric spaces and applied it to ordinary differential equation. The existance and uniqueness of a solution of differential and integral equation is nothing but the existance and uniqueness of a fixed point of a mapping operator. To study the existance and uniqueness of a solution of differential equation, we converted differential equation into integral equation and applied the approximation approach to get the exact solution.

2 Preliminaries

2.1 Fixed point:

Let f be a mapping from X to itself. Then fixed point of a mapping f is a point $x \in X$ such that $f(x) = x$

Fixed point of a mapping ensures the existance of solution of an equation.

2.2 Banach Contraction Principle (BCP):

Let (X, d) be the metric space then $f : X \to X$ is said to be a Lipschitz continuous if there exists $\lambda \geq 0$, such that $d[f(x), f(y)] \leq \lambda f(x, y)$, $\forall x, y \in X$, where λ is called Lipschitz constant.

If $\lambda \leq 1$ then f is called non expansive and if $\lambda < 1$ then f is called contraction mapping. Let f be a contraction mapping on complete metric space X, then f has unique fixed point $x \in X$ such that $f(x) = x$

2.3 ELZAKI Transform:

[5]

Elzaki transform which is modification of Laplace and SUMUDU transform. It helps in solving Linear differential equation, Partial differential equation, ODE, Integral equations and system of equations easily

Many differential and integral equations can be solved by Elzaki transform which are not easily solvable by Laplace or Sumudu transform.

Elzaki transform of a function belongs to class A, where

$$A = [f(t)| \ \exists \ M, k_1, k_2 > 0 \ such \ that \ |f(t)| < Me^{\frac{|t|}{k_j}}]$$

$if\ t \in (-1)^j X[0, \infty]$

Where $E[f(t)] = T(u)$, Elzaki transform of $f(t)$ and is given by an integral as

$E[f(t)] = T(u) = u^2 \int_0^\infty f(ut)e^{-t}dt, u \in (k_1, k_2)$

2.3.1 Some standard results

1. $E[t] = u^3$ and $L[t] = \dfrac{1}{s^2}$

2. $E[1] = u^2$ and $L[1] = \dfrac{1}{s}$

3. $E[sinat] = \dfrac{au^3}{1 + a^2u^2}$ and $L[sinat] = \dfrac{a}{s^2 + a^2}$

4. $E[cosat] = \dfrac{u^2}{1 + a^2u^2}$ and $L[cosat] = \dfrac{s}{s^2 + a^2}$

5. $E[e^{at}] = \dfrac{u^2}{1 - au}$ and $L[e^{at}] = \dfrac{1}{s - a}$

6. $E[t^n] = n!s^{n+1}$ and $L[t^n] = \frac{\gamma(n+1)}{s^{n+1}}$

2.3.2 Elzaki transform of derrivative:[4]

$E[y'(t)] = \frac{T(u)}{u} - uy(0)$

$E[y''(t)] = \frac{T(u)}{u^2} - y(0) - uy'(0)$

$E[y'''(t)] = \frac{T(u)}{u^3} - \frac{y(0)}{u} - y'(0) - uy''(0)$

$\vdots$

$$E[y^n(t)] = \frac{T(u)}{u^n} - \sum_{k=0}^{n-1}[u^{2-n+k}y^k(0)]$$

3 Main Result:

3.1 Solution of Integral equation:

Let $K_1, K_2 : [a \times b] \times [a \times b] \times R \to R$ and $h : [a, b] \to R$ be the given continuous functions.

$$f(t) = \alpha A(t, f(t)) + \beta B(t, f(t)) + h(t), t \in [a, b] \tag{3.1}$$

Where $A(t, f(t) = \int_a^t K_1(t, s, f(s))ds$ *and* $A(t, f(t) = \int_a^t K_2(t, s, f(s))ds, \alpha, \beta \in \mathbb{R}$

To have non trivial solution of (3.1), we have $|\alpha| + |\beta| > 0$.

Now to discuss the existence and uniqueness of solution of above integral equation, let us consider following conditions.

Let C be the space of all real valued continuous function on $I = [a, b]$

Consider the Beilecki norm as follows:[2]

$||f||_B = \sup_{t \in I} |f(t)|e^{-\theta t}, \quad f \in C(I, R), \quad \theta > 0 \ (fixed)$

Let us define the metric d on $C(I, R)$ as

$$d(u, v) = Sup_{t \in I}|u(t) - v(t)|e^{-\theta t}, \ for\ all\ u, v \in C(I, R) \tag{3.2}$$

If T be a mapping from $C(I, R)$ to itself defined as

$$T(f(t)) = \alpha A(t, f(t)) + \beta B(t, f(t)) + h(t), t \in I = [a, b] \tag{3.3}$$

Now if $T(f(t)) = f(t)$ then (3.1) has solution. Now we find the fixed point of T.

Theorem 3.1. *Let* $K_1, K_2 : I \times I \times R \to R$ *are continuoue function and* $h : I \to R$ *and for all* $s, t \in I, u, v \in R$ $\exists L_1, L_2 : I \times I \times R \times R \to R^+$

where L_1, L_2 *are continuous function such that*

$$|K_i(t,s,u(s)) - K_i(t,s,v(s))| \leq L_i(t,s,u(s)v(s))|u(s)-v(s)| \tag{3.4}$$
$$\leq \gamma|u(s)-v(s)|$$

Where $\gamma > 0$ *such that* $L_i(t,s,u(s),v(s)) \leq \gamma, i = 1,2$ *then (3.1) has unique solution in* C *.*

Proof. We have from (3.3)

$T(f(t)) = \alpha A(t,f(t)) + \beta B(t,f(t)) + h(t), t \in I = [a,b]$

To show that T ahs fixed point , it is suficient to show that T is a contraction mapping on C .

As C is a set of real valued continuous function on I , obviously it is complete metric space.

$$\begin{aligned}
d(Tu,Tv) &= Tu(t) - Tv(t) \\
&= \sup_{t\in I}\Big|\alpha A(t,u(t)) + \beta B(t,u(t)) - \alpha A(t,v(t)) - \beta B(t,v(t))\Big|e^{-\theta t} \\
&= \sup_{t\in I}\Big|\alpha\int_a^t [K_1(t,s,u(s)) - K_1(t,s,v(s))]ds + \beta\int_0^t [K_2(t,s,u(s)) - K_2(t,s,v(s))]ds\Big|e^{-\theta t} \\
&\leq \sup_{t\in I}\left[|\alpha|\int_a^t |K_1(t,s,u(s)) - K_1(t,s,v(s))|ds + |\beta|\int_0^t |K_2(t,s,u(s)) - K_2(t,s,v(s))|ds\right]e^{-\theta t} \\
&\leq \sup_{t\in I}\left[|\alpha|\int_a^t L_1(t,s,u(s),v(s))\ |u(s)-v(s)|ds + |\beta|\int_0^t L_2(t,s,u(s),v(s))\ |u(s)-v(s)|ds\right]e^{-\theta t}
\end{aligned}$$

but from equation (3.2), $\frac{d(u,v)}{e^{-\theta s}} = \sup_{t\in I} |u(s) - v(s)|$

$$\leq \sup_{t\in I} \int_a^t [|\alpha| L_1(t,s,u(s),v(s)) + |\beta| L_2(t,s,u(s),v(s))]\, e^{\theta s} e^{-\theta t} ds$$

$$\leq d(u,v) \sup_{t\in I} \int_a^t [|\alpha|\gamma + |\beta|\gamma]\, e^{(s-t)\theta} ds$$

$$= d(u,v)\gamma\,[|\alpha| + |\beta|] \int_a^t e^{(s-t)\theta} ds$$

$$= d(u,v)\gamma\,[|\alpha| + |\beta|] \left[\frac{e^{(s-t)\theta}}{\theta}\right]_a^t$$

$$= d(u,v)\gamma\,[|\alpha| + |\beta|] \left[\frac{1 - e^{-(a-t)\theta}}{\theta}\right]$$

$$= d(u,v)\gamma \frac{[|\alpha| + |\beta|]}{\theta} \left[1 - e^{-(a-t)\theta}\right]$$

Since θ is arbitrary, let $\theta = \gamma\,[|\alpha| + |\beta|]$

$= d(u,v)\left[1 - e^{-(a-t)\theta}\right]$

$= \lambda d(u,v)$, where $\lambda = \left[1 - e^{-(a-t)\theta}\right] < 1$

$\Longrightarrow$ T is contraction mapping over C, since C is complete metric space , by Banach contraction mapping principle, it has unique fixed point, which is unique solution of integral equation (3.1) in C(I, R). □

Similarly if we consider **Fredholm integral equation** of second kind as follows:

$$f(t) = \alpha A(t, f(t)) + \beta B(t, f(t)) + h(t), t \in [a,b]$$

Where $A(t, f(t) = \int_a^b K_1(t,s,f(s))ds$ *and* $A(t, f(t) = \int_a^b K_2(t,s,f(s))ds, \alpha, \beta \in \mathbb{R}$

To have non trivial solution of (3.1), we have $|\alpha| + |\beta| > 0$.

Now to discuss the existence and uniqueness of solution of above integral equation, let us consider following conditions.

Let C be the space of all real valued continuous function on $I = [a,b]$

Consider the Beilecki norm as follows:

$||f||_B = \sup_{t\in I} |f(t)|e^{-\theta t}, \quad f \in C(I,R), \quad \theta > 0 \ (fixed)$

Let us define the metric d on $C(I,R)$ as

$$d(u,v) = Sup_{t\in I}|u(t) - v(t)|e^{-\theta t}, \ for\ all\ u,v \in C(I,R)$$

If T be a mapping from $C(I,R)$ to itself defined as

$$T(f(t)) = \alpha A(t, f(t)) + \beta B(t, f(t)) + h(t), t \in I = [a,b]$$

Now if $T(f(t)) = f(t)$ then (3.1) has a solution.

Example 3.2. *Solve* $\frac{du}{dt} = u + 1, u(0) = 5$

Solution: *First we solve this initial value problem by Elzaki transform method and then by Banach contraction principle.*

we have $y'(t) = y(t) + 1$, *Taking Elzaki transform we get,*

$$\frac{T(u)}{u} - uy(0) = T(u) + u^2$$

$$T(u)\left[\frac{1}{u} - 1\right] = u^2 + 5u$$

$$T(u) = (u^2 + 5u)(\frac{u}{1-u})$$

$$= \frac{5u^2 + u^3}{1-u}$$

$$= \frac{6u^2}{1-u} - u^2$$

Taking inverse Elzaki transform we get, $y(t) = 6e^t - 1$,

Which is a solution of given IVP.

Now we solve above IVP by BCP method as follows:

Where $K = u(t) + 1, L(s, u(s), v(s)) = 1 = \alpha$

$\left|K(s, u(s) - K(s, v(s))\right| = |u + 1 - v - 1| = |u - v|$

Which satisfies all the condition of theorem (3.1) therefore this IVP has unique solution which is sequence $\{z_n\}$ *with* $z_n = Tz_{n-1}$

To solve this, first we convert given diiferential equation into integral equation by integrating from t_0 *to* t *, we get*

$T(u(t)) = u(0) + \int_0^t [u(s) + 1]ds$ *we start with* $z_0 = u(0) = 5$

$z_1(t) = T[z_0(t)] = 5 + \int_0^t [5 + 1]ds = 5 + 6t$

$z_2(t) = T[z_1(t)] = 5 + \int_0^t [6 + 6s]ds = 5 + 6t + 3t^2$

$z_3(t) = T[z_2(t)] = 5 + \int_0^t [6 + 6s + 3s^2]ds = 5 + 6t + 3t^2 + t^3$

Similarly

$z_4(t) = 5 + 6t + 3t^2 + t^3 + \frac{t^4}{4}$

$\vdots$

$$\begin{aligned} z_n(t) &= 5 + 6t + 3t^2 + t^3 + \frac{t^4}{4} + \frac{t^5}{20} + \dots \\ &= 5 + 6[(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots) - 1] \\ &= 5 + 6[e^t - 1] \\ &= 6e^t - 1 \end{aligned}$$

Which is a solution of given IVP matching with the exact solution solved by Elzaki transform.

Example 3.3. *Solve* $\frac{du}{dt} = 0.85u, u(0) = 19$

Solution: *We have* $\left|K(s, u(s) - K(s, v(s))\right| = 0.85|u - v|$

$L(s, u(s), v(s)) = 0.85 = \alpha > 0, k(t, u) = 0.85u$

$u(t) = u_0 + \int_0^t 0.85u(s)ds$ *. Let* $z_n = Tz_{n-1}$ *start with* $z_0 = u_0 = 19$

$z_1(t) = T[z_0(t)] = 19 + \int_0^t 0.85 * 19ds = 19[1 + 0.85t]$

$z_2(t) = T[z_1(t)] = 19 + \int_0^t 0.85 * 19[1 + 0.85s]ds = 19[1 + 0.85t + \frac{(0.85t)^2}{2}]$

$z_3(t) = T[z_2(t)] = 19 + \int_0^t 0.85 * 19[1 + 0.85s + \frac{(0.85s)^2}{2}]ds = 19[1 + 0.85t + \frac{(0.85t)^2}{2} + \frac{(0.85t)^3}{3!}]$

$\vdots$

$\{z_n\} = 19e^{0.85t}$ *Which is a solution of given IVP.*

4 Conclusion:

We obtained the solution of initial value problem by the fixed point method. Which is supported by the examples.Which are again compared with the solutions of IVP by Elzaki transform method which is recent development of Tarig M. Elzaki, Salih M. Elzaki as extension of Laplace transform method for the same.

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