On The Outer Measure Of Soft Sets

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ABSTRACT

This present research article introduces the outer measure of soft sets. Some interesting results on the properties and the measurability of soft sets are investigated in this work.

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Introduction:

In 1999, Molodtsov [5] introduced the soft set theory as a general mathematical tool for dealing with uncertainty or vagueness. There are many theories like Fuzzy set theory and Rough set theory to tackle the problem of imperfect knowledge. Soft set theory is still a better approach to deal with problems involving uncertainty. Molodtsov recognized the importance of the role of parameters and introduced the theory of Soft sets. He has shown several applications of this theory in many fields like economics, engineering, medical sciences, etc. Later, this theory became a very good source of research for many mathematicians and computer scientists of recent years because of its wide range of applicability. The development in the fields of soft set theory and its application has been taking place in a rapid pace.

The notion of soft groups is a branch of soft set theory and it was first introduced by Aktas and Cagman in 2007. Later, some authors like Yin and Liao have studied various properties of soft groups. The soft group theory has applications in the theory of computer science. The soft set theory is used in topology and soft topologies for also introduced. Many authors probed deeply into this area and proposed several concepts.

In this present work, we define an outer measure of a soft set by making use of the Lebesgue outer measure of a subset of the real line. We establish a few results in this context. We also introduce the notion of a measurable soft set.

In what follows \( \mathbb{R} \) and \( P(\mathbb{R}) \) stand for the real line and the collection of all subsets of \( \mathbb{R} \) respectively. We denote the empty set by using the symbol \( \emptyset \) throughout this Article.

1. Preliminaries

In this section, we present some basic definitions that are needed in further study of this work. Let \( U \) be an initial universe set and \( E_U \) (or simply \( E \)) be a collection of all possible parameters with respect to \( U \), where parameters are the characteristics or properties of objects in \( U \). Let \( P(U) \) be the collection of all subsets of \( U \).

**Definition-1.1:** A pair \( (F, A) \) is called a soft set over \( U \), if \( A \subseteq E \) and \( F : A \rightarrow P(U) \). We write \( F_A \) for \( (F, A) \).

**Definition-1.2:** Let \( F_A \) and \( G_B \) be soft sets over a common universe set \( U \) and \( A, B \subseteq E \). Then we say that

- \( F_A \) is a soft subset of \( G_B \), denoted by \( F_A \sqsubseteq G_B \), if
  - (a) \( A \subseteq B \) and \( e \in A \)
  - (b) \( F_A \) equals \( G_B \), denoted by \( F_A \simeq G_B \), if \( F_A \sqsupseteq G_B \) and \( G_B \sqsubseteq F_A \).

**Definition-1.3:** A soft set \( F_A \) over \( U \) is called a null soft set, denoted by \( \Phi \), if \( e \in A \), \( F(e) = \emptyset \).

**Definition-1.4:** A soft set \( F_A \) over \( U \) is called an absolute soft set, denoted by \( F_A \), if \( e \in A \), \( F(e) = U \).
Definition-1.5: The union of two soft sets $F_A$ and $G_B$ over a common universe $U$ is the soft set $H_C$, where $C = A \cup B$, and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A - B \\ G(e) & \text{if } e \in B - A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$$

We write $F_A \cup G_B = H_C$.

Definition-1.6: The intersection of two soft sets $F_A$ and $G_B$ over a common universe $U$ is the soft set $H_C$, where $C = A \cap B$, and for all $e \in C$, $H(e) = F(e) \cap G(e)$.

We write $F_A \cap G_B = H_C$.

Definition-1.7: For a soft set $F_A$ over $U$, the relative complement of $F_A$ is denoted by $F_A^c$ and is defined by $F_A^c = F_A^l$ where $F^1 : A \to P(U)$ is a mapping given by $F^1(e) = U - F(e)$ for all $e \in A$.

Definition-1.8: If $F_A$ and $G_B$ are two soft sets over $U$, then $F_A$ AND $G_B$ is denoted by $F_A \land G_B$ and it is defined as a soft set $(H, A \times B)$ where

$$H(\alpha, \beta) = F(\alpha) \cap G(\beta), \forall (\alpha, \beta) \in A \times B.$$ 

Definition-1.9: If $F_A$ and $G_B$ are two soft sets over $U$, then $F_A$ JOIN $G_B$ is denoted by $F_A \lor G_B$ and it is defined as $(H, A \times B)$ where

$$H(\alpha, \beta) = F(\alpha) \lor G(\beta), \forall (\alpha, \beta) \in A \times B.$$ 

Definition-1.10: Suppose a binary operation denoted by $\circ$, is defined for all subsets of $U$. Let $F_A$ and $G_B$ be two soft sets over $U$. Then the operation $\circ$ for the soft sets is defined in the following way: $F_A \circ G_B = (H, A \times B)$ Where

$$H(\alpha, \beta) = F(\alpha) \circ G(\beta), \alpha \in A \text{ and } \beta \in B.$$ 

2. Outer measure of a soft set

This section is devoted to define the outer measure of a soft set. A few interesting results are presented in this context. First, we present the definition of outer measure of a subset of $\mathbb{R}$ in the sense of Lebesgue [6] and we make use of this notion to introduce our concept of the outer measure of a soft set. We call this concept by the name, the soft outer measure.

Definition-2.1 [6]: The length of an interval $I$ on $\mathbb{R}$ is defined to be the difference between its end points and it is non-negative. It is denoted by $l(I)$.

The outer measure of a subset $A$ of $\mathbb{R}$ in the sense of Lebesgue is denoted by $m^*A$ and it is defined by

$$m^*A = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$$

Where the infimum runs over all sequences $\{I_n\}$ of open intervals with $A \subseteq \bigcup_{n=1}^{\infty} I_n$.

Remarks-2.2: The following facts on the Lebesgue outer measure are well known

1. $m^*\emptyset = 0$
2. The outer measure of a finite set is zero.
3. The outer measure of an interval is its length.
4. The Lebesgue outer measure is monotonic. That is,
The outer measure is countably subadditive. That is, if \( \{E_n\} \) is any sequence of subsets of \( \mathbb{R} \) then

\[
m^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m^*(E_n)
\]

The outer measure is translation invariant. That is

\[
m^*(E + y) = m^*E \quad \text{where} \quad E + y = \{x + y : \ x \in E\}
\]

In order to define the outer measure of a soft set or the soft outer measure, we take the universe set to be the real line \( \mathbb{R} \). The soft sets that we consider are defined on some set of parameters denoted by \( E \). That is, our soft sets are mappings \( F : E \to P(\mathbb{R}) \), where \( E \) is some set of parameters and these soft sets are denoted by \( F_E \).

**Definition-2.3:** Let \( F : E \to P(\mathbb{R}) \) be any soft set. We denote the outer measure of \( F_E \) or the soft outer measure of \( F_E \) by the symbol \( s^*F_E \) and we define it as follows.

\[
s^*F_E = \inf_{e \in E} m^*(F(e))
\]

**Remark-2.4:** The proof of the following Proposition-2.5 is easy to verify and hence can be omitted.

**Proposition-2.5:** If \( \Phi : E \to P(\mathbb{R}) \) is the null soft set defined by \( \Phi(e) = \emptyset \ \forall \ e \in E \) and \( \mathcal{U} : E \to P(\mathbb{R}) \) is the absolute soft set defined by \( \mathcal{U}(e) = \mathbb{R} \ \forall \ e \in E \) then

(i) \( s^*\Phi_E = 0 \)

(ii) \( s^*\mathcal{U}_E = +\infty \)

**Proposition-2.6:** If \( F_E \) and \( G_E \) are any two soft sets such that \( F_E \subseteq G_E \) then \( s^*F_E \leq s^*G_E \)

**Proof:** Since \( F_E \subseteq G_E \), we have \( F(e) \subseteq G_e \ \forall \ e \in E \) then

\[
m^*(F(e)) \leq m^*(G(e)) \ \forall \ e \in E
\]

\[
\Rightarrow s^*F_E = \inf_{e \in E} m^*(F(e)) \leq m^*(F(e)) \leq m^*(G(e)) \ \forall \ e \in E
\]

\[
\Rightarrow s^*F_E \leq s^*G_E
\]

Thus \( s^* \) is monotonic.

**Definition-2.7:** If \( \{F_E^{(n)}\} \) is a sequence of soft sets with \( F^{(n)} : E \to P(\mathbb{R}) \), \( n = 1, 2, 3,... \) then we define the soft set \( H_E = \bigcup_{n=1}^{\infty} F_E^{(n)} \) by \( H(e) = \bigcup_{n=1}^{\infty} F^{(n)}(e) \ \forall \ e \in E \)

**Proposition-2.8:** If \( \{F_E^{(n)}\} \) is a sequence of soft sets with \( F^{(n)} : E \to P(\mathbb{R}) \), \( n = 1, 2, 3,... \) then

\[
s^*\left( \bigcup_{n=1}^{\infty} F_E^{(n)} \right) \leq \inf_{e \in E} \sum_{n=1}^{\infty} m^*\left( F^{(n)}(e) \right)
\]

**Proof:** Let \( H_E = \bigcup_{n=1}^{\infty} F_E^{(n)} \). Then \( s^*H_E = s^*\left( \bigcup_{n=1}^{\infty} F_E^{(n)} \right) \).
\[
\begin{align*}
\text{Definition-2.9:} & \text{ We say that a soft set } F : E \to P(\mathbb{R}) \text{ is} \\
& \quad (a) \text{ a soft single point set if } F(e) \text{ is a single point set for all } e \in E. \text{ That is there exists a point } \\
& \quad \quad x \in \mathbb{R} \text{ such that } F(e) = \{x\} \forall e \in E. \\
& \quad (b) \text{ a soft singleton, if each } F(e) \text{ is a single point set in } \mathbb{R}. \\
& \quad (c) \text{ a soft countable set if } F(e) \text{ is a countable set for every } e \in E. \\
& \quad (d) \text{ a countable soft set if each } F(e) \text{ is countable in } \mathbb{R}. \\

\text{Proposition-2.10:} & \text{ If } F_E \text{ is any one of the above defined soft set then } s^* F_E = 0
\end{align*}
\]

3. Soft measurability
In this section, we introduce the concept of soft measurable sets.

**Definition-3.1:** We say that a soft set \( F_E \) is soft measurable if for any soft set \( G_E \), we have
\[
s^*(G_E) \geq s^*(F_E \cap G_E) + s^*(G_E \cap F_E^c)
\]

**Proposition-3.2:** If \( s^* F_E = 0 \) then \( F_E \) is soft measurable.

**Proof:** Since \( G_E \cap F_E \subseteq F_E \) and \( s^* F_E = 0 \), we have \( s^*(G_E \cap F_E) = 0 \) for any soft set \( G_E \).

Since \( G_E \cap F_E^c \subseteq G_E \), \( s^*(G_E \cap F_E^c) \leq s^* G_E \)

\[
\Rightarrow s^*(G_E) \geq s^*(F_E \cap G_E) + s^*(G_E \cap F_E^c)
\]

\[
\Rightarrow F_E \text{ is soft measurable.}
\]

**Definition-3.3 [6]:** A subset \( B \) of \( \mathbb{R} \) is said to be Lebesgue measurable if for any subset \( A \) of \( \mathbb{R} \),
\[
m^*(A) = m^*(A \cap B) + m^*(A \cap B^c)
\]

where \( B^c = \mathbb{R} - B \)

**Proposition-3.4:** If \( F(e) \) is Lebesgue measurable for each \( e \in E \) then \( F_E \) is soft measurable.

**Proof:** Since \( F_E \) is Lebesgue measurable for each \( e \in E \) there exists a subset \( A \) of \( \mathbb{R} \) such that
\[
m^* A = m^*(A \cap H_e) + m^*(A \cap H_e^c) \quad \text{for all } e \in E \text{ where } H_e = F(e)
\]

Define \( G : E \to P(\mathbb{R}) \) by \( G(e) = A \) for all \( e \in E \)

Then \( m^* G(e) = m^*(G(e) \cap H_e) + m^*(G(e) \cap H_e^c) \quad \text{for all } e \in E \)

\[
\geq \inf_{e \in E} m^*(G(e) \cap H_e) + \inf_{e \in E} m^*(G(e) \cap H_e^c)
\]
\[ s^* \left( G_E \cap F_E \right) + s^* \left( G_E \cap F_E^c \right) \]
\[ \Rightarrow s^* \left( G_E \right) \geq s^* \left( F_E \cap G_E \right) + s^* \left( G_E \cap F_E^c \right) \]
\[ \Rightarrow F_E \text{ is soft measurable.} \]

References

Man is not made for defeat. A man can be destroyed, but not defeated.
~ Ernest Hemingway